Programlama -1

“Fourier Analysis”

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Fourier Analysis

Fourier analizi, bir fonksiyonu sinüzoidal bileşenler cinsinden temsil etme işlemidir. Mühendislik, bilim ve uygulamalı matematiğin birçok alanında yaygın olarak kullanılmaktadır. Hangi frekans bileşenlerinin bir işlevi temsil ettiğine dair bilgi sağlar.

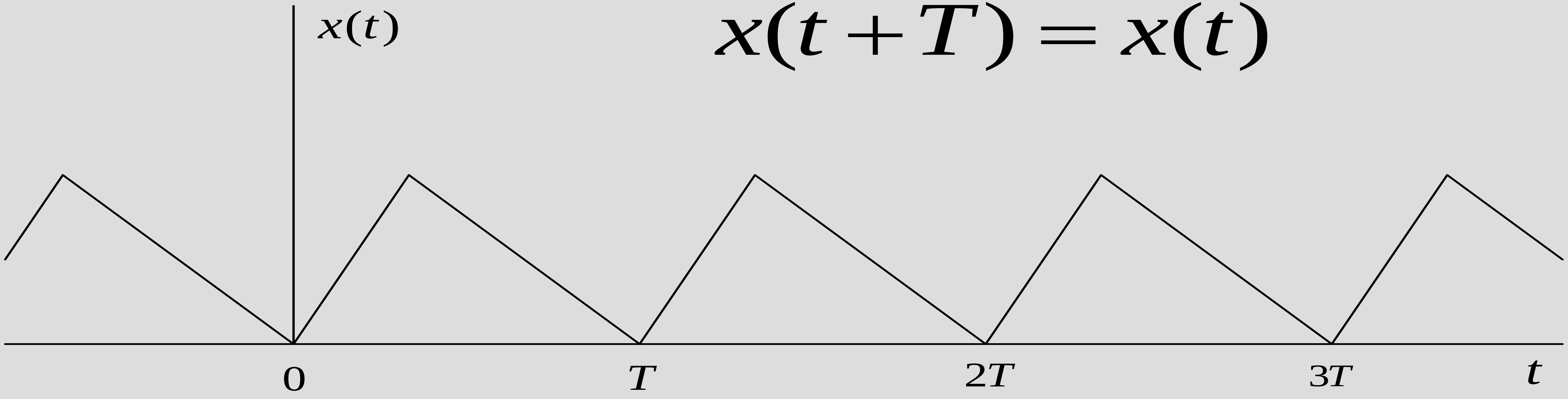
Initial Assumptions

Since many of the variables for which Fourier analysis is required are functions of time t, we will use it as the independent variable. The dependent variable will be denoted as $x(t)$. The function $x(t)$ is said to be in the *time domain* and the Fourier representation is said to be in the *frequency domain*. The frequency domain form is called the *spectrum*. We will first consider the spectra of *periodic functions.* A periodic function satisfies $x(t) = x(t+T)$

Periodic Function

The function below is an example of a periodic function with period *T*.

$$x(t+T)=x(t)$$



Fourier Series for Periodic Functions

Spectral analysis of periodic functions is achieved through the *Fourier series.* The 3 forms are

(1) cosine-sine form,

(2) amplitude-phase form, and

(3) complex exponential form.

(1) and (2) are referred to as *one-sided forms* and (3) will be referred to as a *two-sided form.* A constant term in the series is often called the *dc value*.

Simple Initial Analysis

(1) Do the series have a constant or dc term?

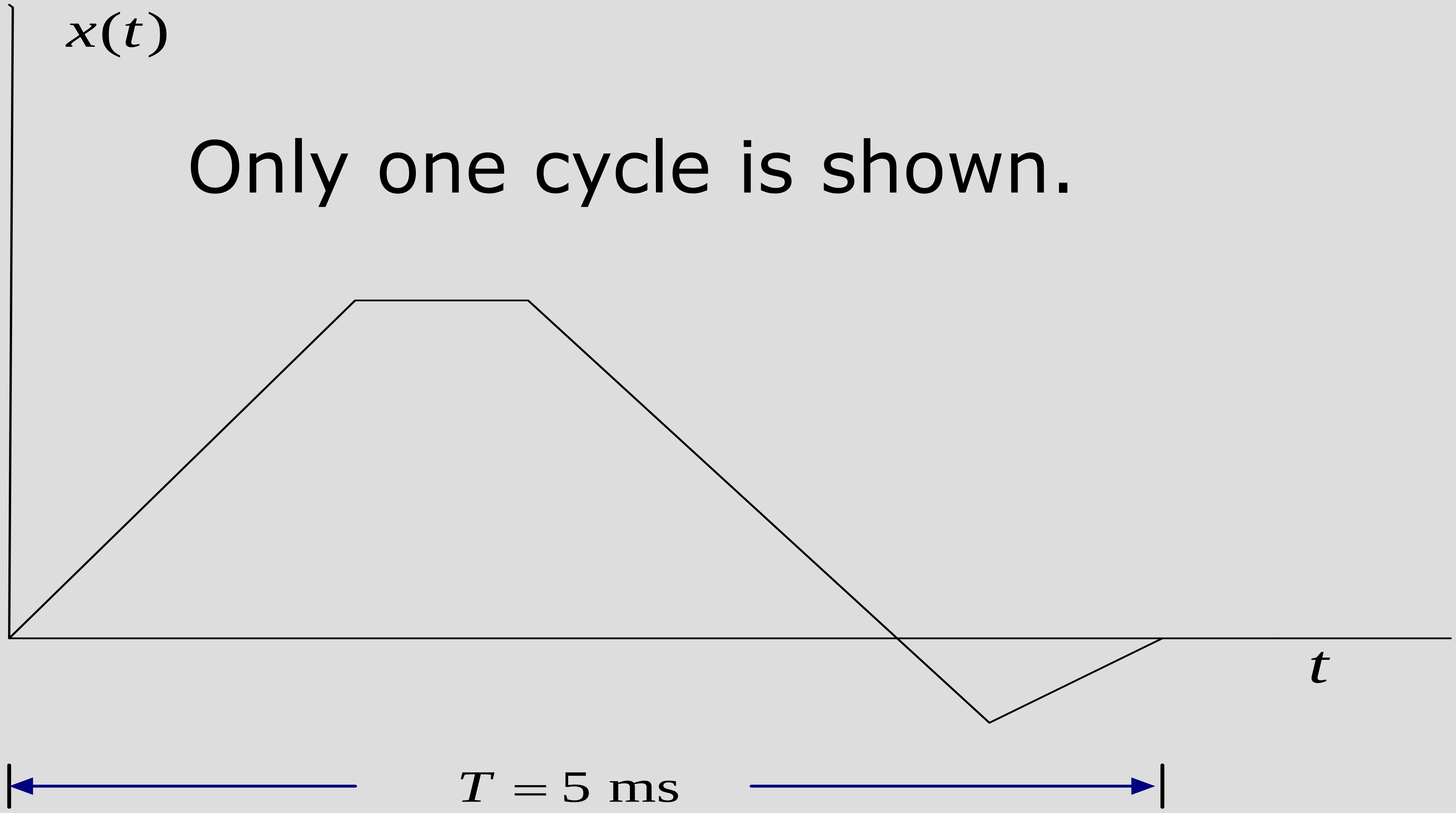
Check to see if the net area in a cycle is 0.

(2) What is the fundamental frequency?

$$f_1 = 1/T$$

(3) All other frequencies are integer multiples of the fundamental; i. e., $2f_1, 3f_1, 4f_1,$ etc.

Example 16-1. List frequencies for the assumed periodic waveform below.



Example 16-1. Continuation.

Since the positive area is clearly greater than the negative area, there will be a constant or dc term in the series.

$$f_1 = \frac{1}{0.005} = 200 \text{ Hz}$$

The frequencies are
0 (dc)
200 Hz
400 Hz
600 Hz
800 Hz, etc.

Fourier Series Cosine-Sine Form

$$x(t) = A_0 + \sum_{n=1}^{\infty} (A_n \cos n\omega_1 t + B_n \sin n\omega_1 t)$$

$$\omega_1 = 2\pi f_1 = \frac{2\pi}{T}$$

$$f_1 = \frac{1}{T}$$

Fourier Series Cosine-Sine Form (Continuation)

$$A_0 = \frac{\text{algebraic area under curve in one cycle}}{T}$$

$$= \frac{1}{T}\int_0^T x(t)dt$$

$$A_n = \frac{2}{T}\int_0^T x(t)\cos n\omega_1 t dt$$

$$B_n = \frac{2}{T}\int_0^T x(t)\sin n\omega_1 t dt$$

Fourier Series Amplitude-Phase Form

$$x(t) = C_0 + \sum_{n=1}^{\infty} C_n \cos(n\omega_1 t + \theta_n)$$

$$C_n = \sqrt{A_n^2 + B_n^2}$$

$$\theta_n = \tan^{-1}\left(\frac{-B_n}{A_n}\right)$$

Fourier Series Complex Exponential Form

$$x(t) = \sum_{n=-\infty}^{\infty} \mathbf{X_n} e^{in\omega_1 t}$$

$$\mathbf{X_n} = \frac{1}{T}\int_0^T x(t)e^{-in\omega_1 t}dt$$

Some Relationships

$$\mathbf{X_{-n}} = \bar{\mathbf{X}}_{\mathbf{n}}$$

$$\mathbf{X_n} = \frac{A_n - iB_n}{2}$$

$$X_n = X_{-n} = \frac{C_n}{2} \qquad \text{for } n \neq 0$$

$$X_0 = C_0$$

Example 16-2. A Fourier series is given below. List frequencies and plot the one-sided amplitude spectrum.

$$x(t) = 12 + 9\cos(2\pi \times 10t + \pi/3)$$
$$+ 6\cos(2\pi \times 20t - \pi/6)$$
$$+ 4\cos(2\pi \times 30t + \pi/4)$$

The frequencies are

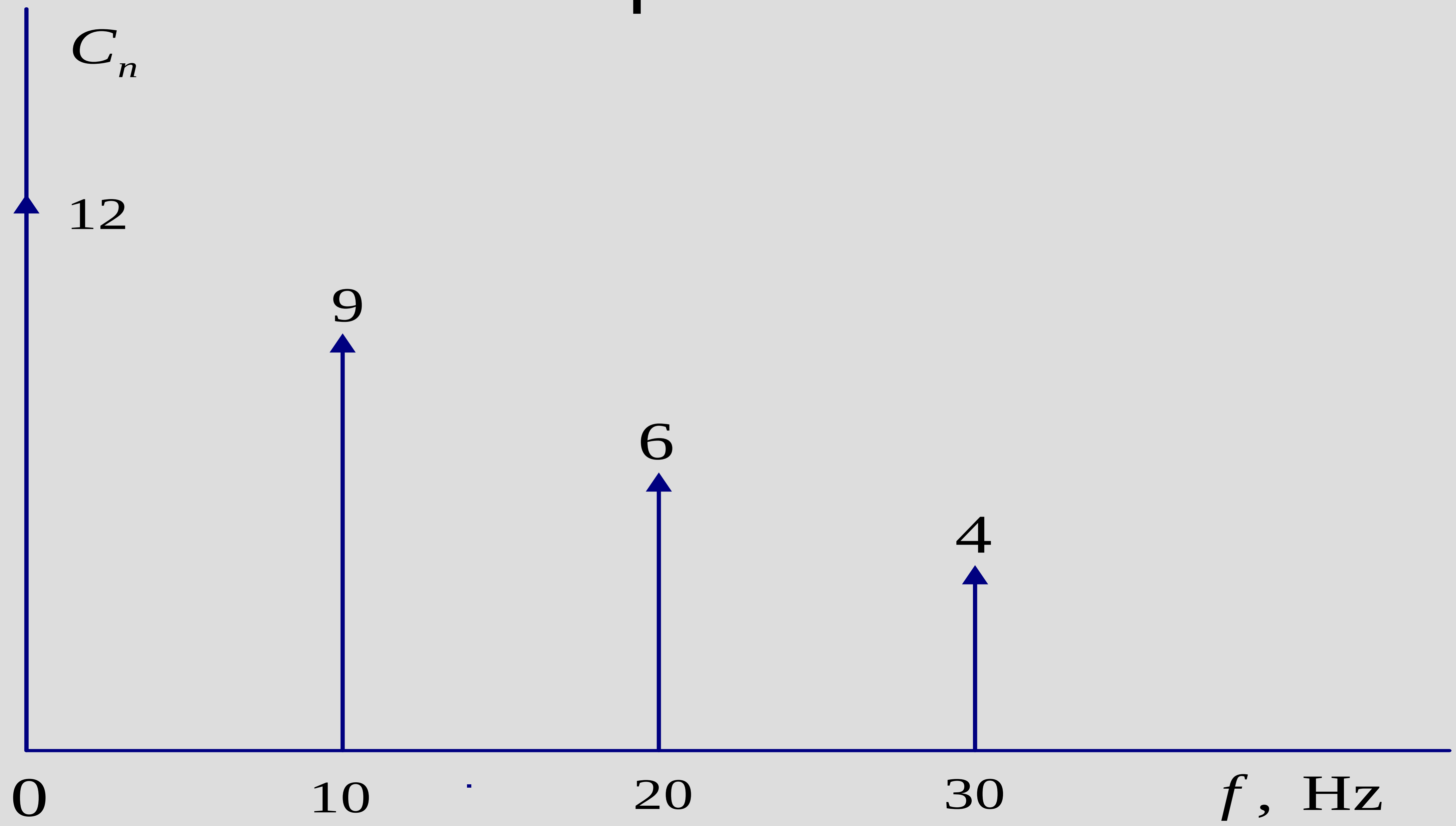
0 (dc)

10 Hz

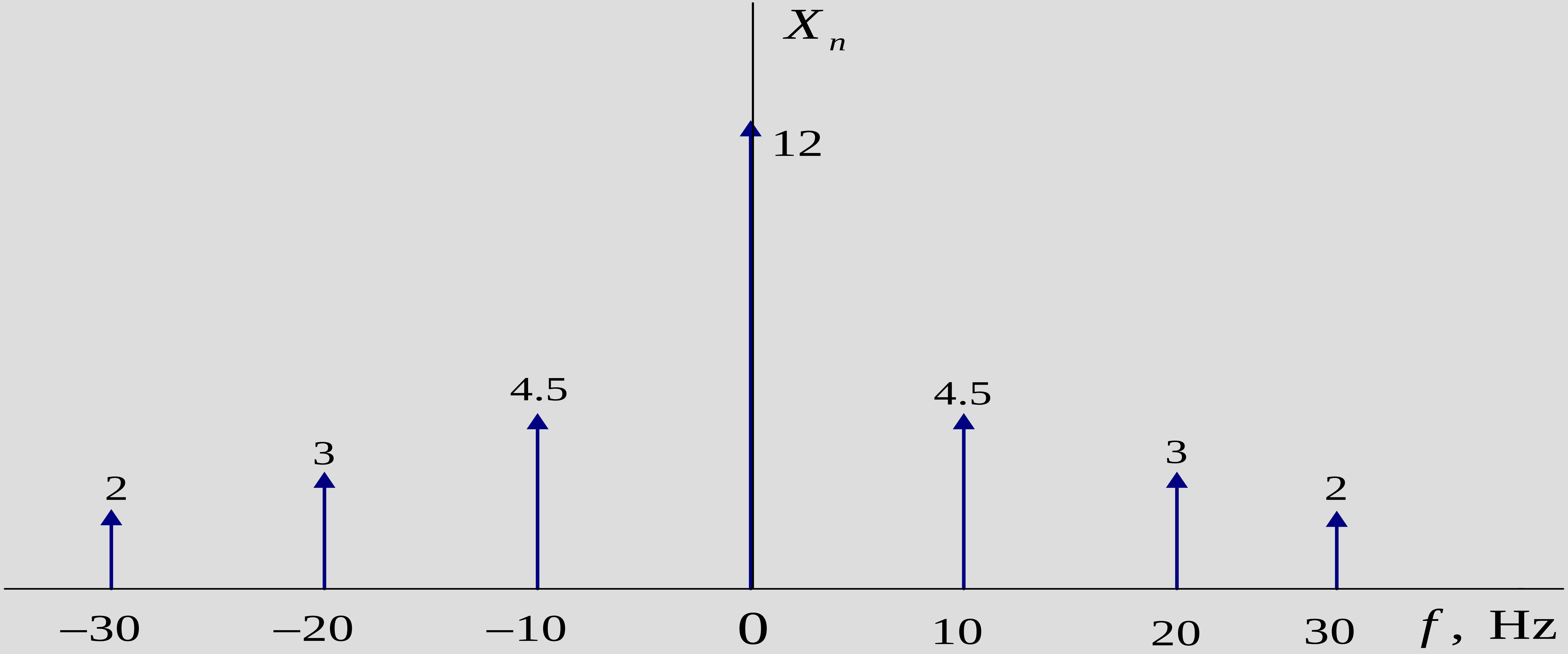
20 Hz

30 Hz

One-Sided Amplitude Spectrum of Example 16-2



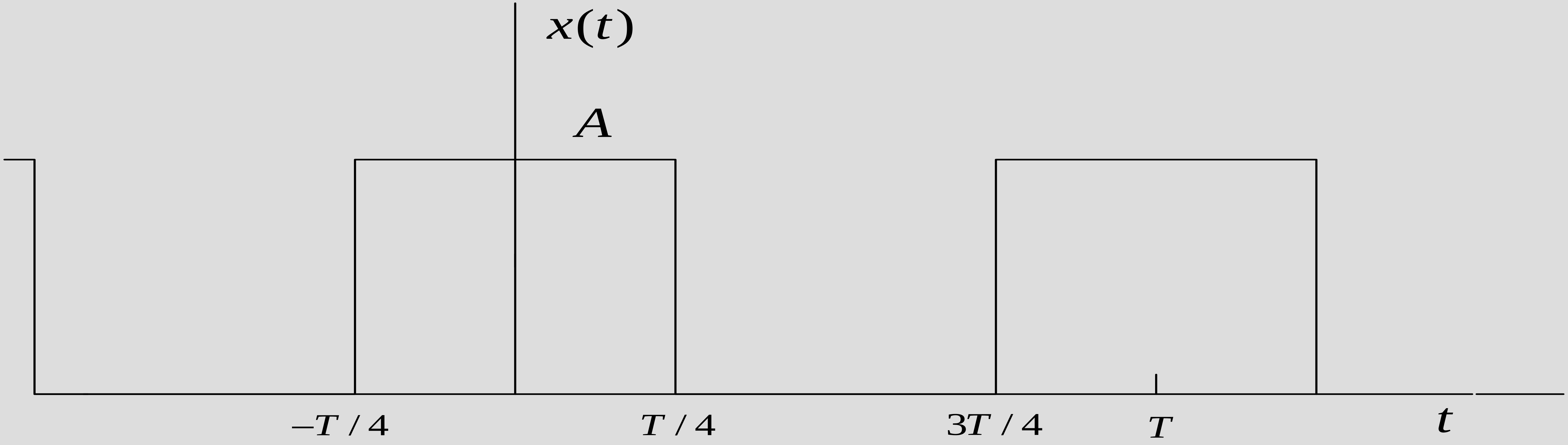
Two-Sided Amplitude Spectrum of Example 16-3



Periodic Function of Examples 16-4, 16-5, and 16-6.

$$x(t) = A \quad \text{for} \quad -T/4 < t < T/4$$

$$= 0 \quad \text{elsewhere in a cycle}$$



Example 16-4. Determine Fourier series.

By inspection, $A_0 = 0.5A$

$$A_n = \frac{2}{T}\int_{-T/2}^{T/2} x(t)\cos n\omega_1 t dt$$

$$= \frac{2}{T}\int_{-T/4}^{T/4} A\cos n\omega_1 t dt$$

$$A_n = \frac{2A}{n\omega_1 T}\left[\sin n\omega_1 t\right]_{-T/4}^{T/4}$$

$$= \frac{2A}{n\omega_1 T}[\sin n\omega_1 T/4 - \sin(-n\omega_1 T/4)]$$

Example 16-4. Continuation.

$$\omega_1 T = 2\pi f_1 T = 2\pi(1/T)T = 2\pi$$

$$A_n = \frac{2A}{n\pi}\sin\frac{n\pi}{2}$$

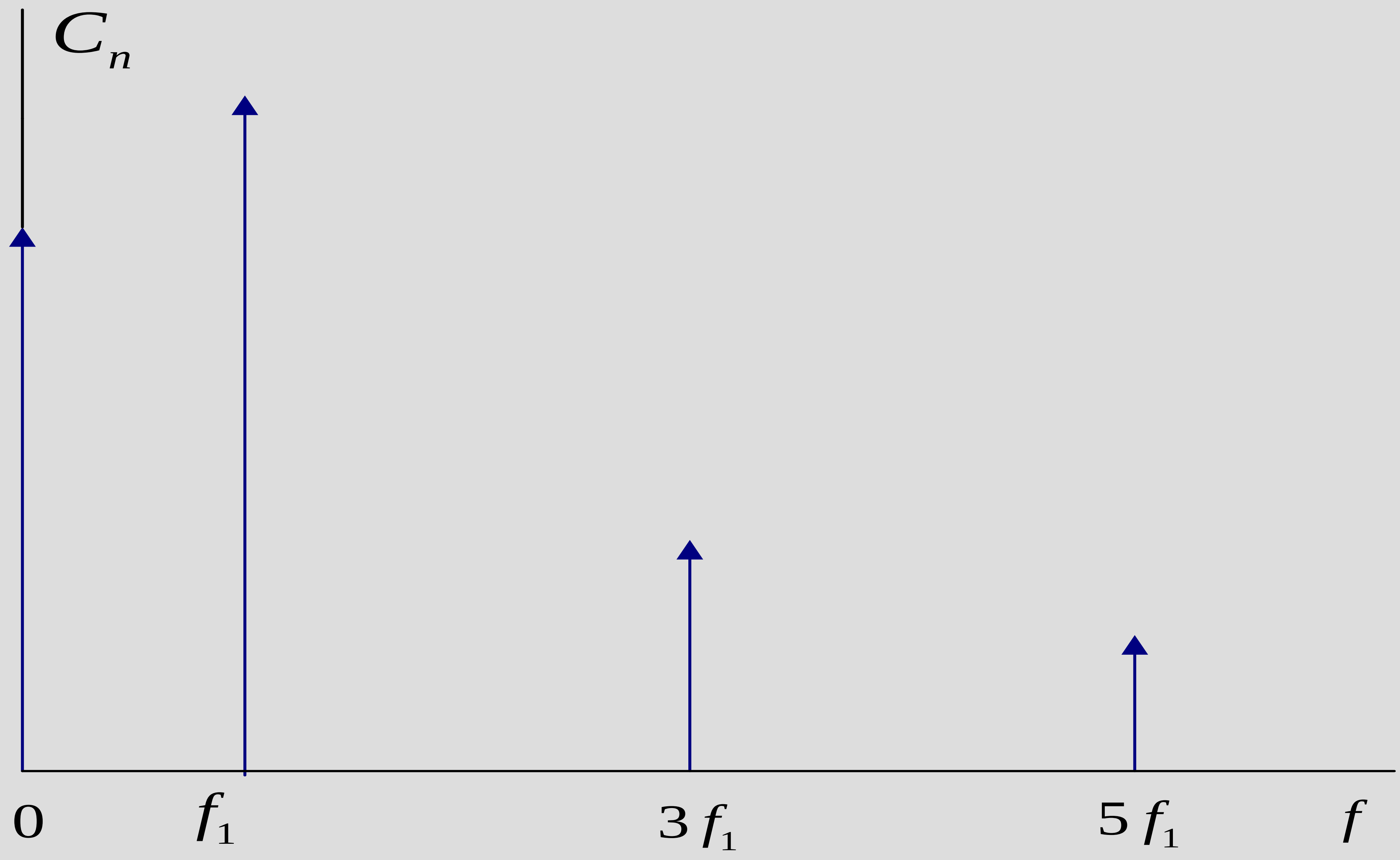
$$\sin\frac{n\pi}{2} = 0 \quad \text{for} \quad n \text{ even}$$

$$= 1 \quad \text{for } n = 1,\ 5,\ 9, \text{ etc.}$$

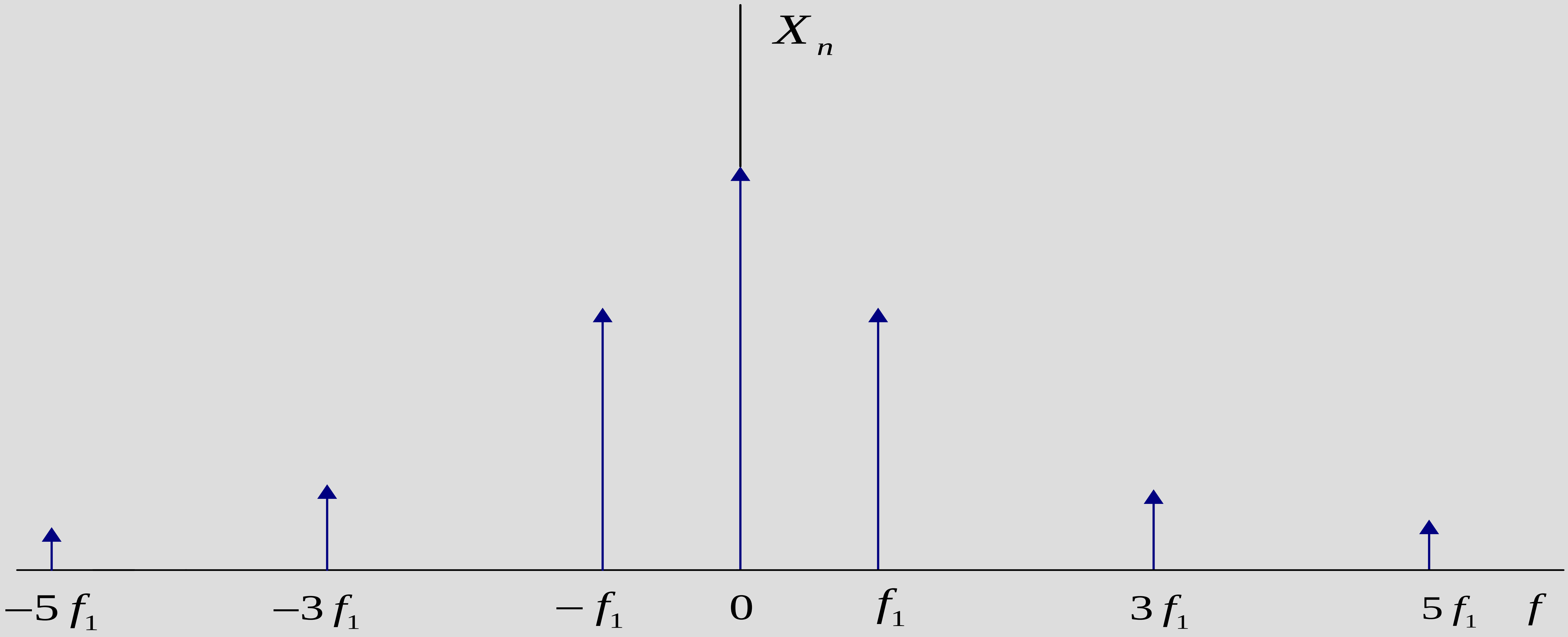
$$= -1 \quad \text{for } n = 3,\ 7,\ 11, \text{ etc.}$$

It can be shown that $B_n = 0$.

Example 16-5. One-Sided Spectral Plot



Example 16-6. Two-Sided Spectral Plot



Fourier Transform

Assume a non-periodic function $x(t)$.

The Fourier transform is $\mathbf{X}(f)$.

$$\mathbf{X}(f) = \int_{-\infty}^{\infty} x(t) e^{-i\omega t} dt$$

$$\omega = 2\pi f$$

$$x(t) = \int_{-\infty}^{\infty} \mathbf{X}(f) e^{i\omega t} df$$

Amplitude and Phase Spectra

$$\mathbf{X}(f) = X(f)e^{i\theta(f)} \square X(f)\angle\theta(f)$$

$$\text{Amplitude Spectrum} = X(f) = \left|\mathbf{X}(f)\right|$$

$$\text{Phase Spectrum} = \theta(f) = \text{ang}\left[\mathbf{X}(f)\right]$$

Comparison of Fourier Series and Fourier Transform

- The Fourier series is usually applied to a periodic function and the transform is usually applied to a non-periodic function.
- The spectrum of a periodic function is a function of a discrete frequency variable.
- The spectrum of a non-periodic function is a function of a continuous frequency variable.

Example 16-7. Determine the Fourier transform of the function below.

$$x(t) = e^{-\alpha t} \quad \text{for} \quad t \geq 0$$
$$= 0 \quad \text{for} \quad t < 0$$

$$\mathbf{X}(f) = \int_{-\infty}^{\infty} x(t) e^{-i\omega t} dt$$
$$= \int_{0}^{\infty} e^{-\alpha t} e^{-i\omega t} dt = \int_{0}^{\infty} e^{-(\alpha + i\omega)t} dt$$

Example 16-7. Continuation.

$$\mathbf{X}(f) = \frac{e^{-(\alpha+i\omega)t}}{-(\alpha+i\omega)}\Bigg]_0^\infty$$

$$= 0 - \frac{1}{-(\alpha+i\omega)} = \frac{1}{(\alpha+i\omega)}$$

$$X(f) = |\mathbf{X}(f)| = \left|\frac{1}{\alpha+i\omega}\right| = \frac{1}{\sqrt{\alpha^2+\omega^2}}$$

$$\theta(f) = -\tan^{-1}\frac{\omega}{\alpha}$$

Example 16-8. Determine the Fourier transform of the function below.

$$x(t) = A \quad \text{for} \quad 0 < t < \tau$$

$$= 0 \qquad \text{elsewhere}$$

$$\mathbf{X}(f) = \int_{-\infty}^{\infty} x(t) e^{-i\omega t} dt = \int_0^{\tau} A e^{-i\omega t} dt = \left. \frac{A e^{-i\omega t}}{-i\omega} \right]_0^{\tau}$$

$$= A\left(\frac{e^{-i\omega\tau} - 1}{-i\omega} \right) = A\left(\frac{1 - e^{-i\omega\tau}}{i\omega} \right)$$

Example 16-8. Continuation.

$$\mathbf{X}(f) = A\tau\left(\frac{\sin \pi f \tau}{\pi f \tau}\right)e^{-i\pi f \tau}$$

$$X(f) = A\tau\left(\frac{\sin \pi f \tau}{\pi f \tau}\right)$$

$$\theta(f) = -\pi\tau f$$

Discrete and Fast Fourier Transforms

- The *discrete Fourier transform* (DFT) is a summation that produces spectral terms that may be applied to either periodic or non-periodic functions.
- The *fast Fourier transform* (FFT) is a computationally efficient algorithm for computing the DFT at much higher speeds.
- The *IDFT* and *IFFT* are the corresponding inverse transforms.

Sampled Signal

Assume that N equally spaced samples of a function are taken with a spacing of Δt between samples. Let n represent the independent variable defined over the domain $0 \le n \le N-1$. The total length of the function is $T = N\Delta t$. For a continuous function $x_c(t)$, define a function $x(n)$ as

$$x(n) = x_c(n\Delta t) \quad \text{for} \quad n \text{ an integer}$$
$$= 0 \quad \text{elsewhere}$$

Definitions of DFT and IDFT

$$X(m) = \sum_{n=0}^{N-1} x(n) e^{-i2\pi mn/N}$$

$$\text{for } 0 \le m \le N-1$$

$$x(n) = \frac{1}{N} \sum_{m=0}^{N-1} X(m) e^{i2\pi nm/N}$$

$$\text{time} \Leftrightarrow n$$

$$\text{frequency} \Leftrightarrow m$$

Initial Assumptions for MATLAB

- The function must be interpreted as periodic.
- It is recommended that the number of points N be even.
- The spectrum will also be periodic. It will be unique only at $N/2$ points.
- The integer for a MATLAB indexed variable must start at a value of 1.
- The highest unambiguous frequency corresponds to a MATLAB index of $N/2$.

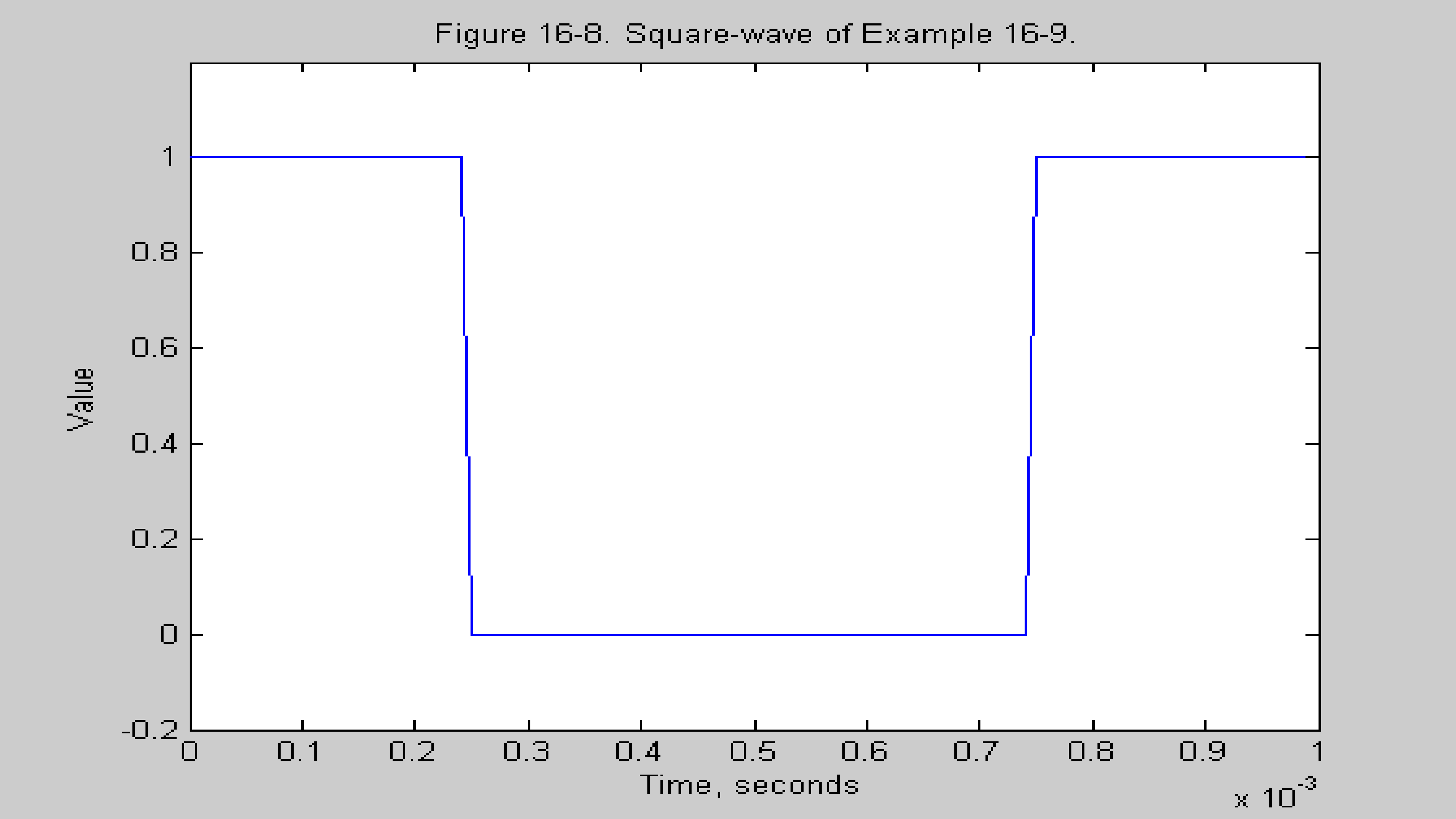


Figure 16-8. Square-wave of Example 16-9.
Value
1
0.8
0.6
0.4
0.2
0
-0.2
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
Time, seconds
x 10-3

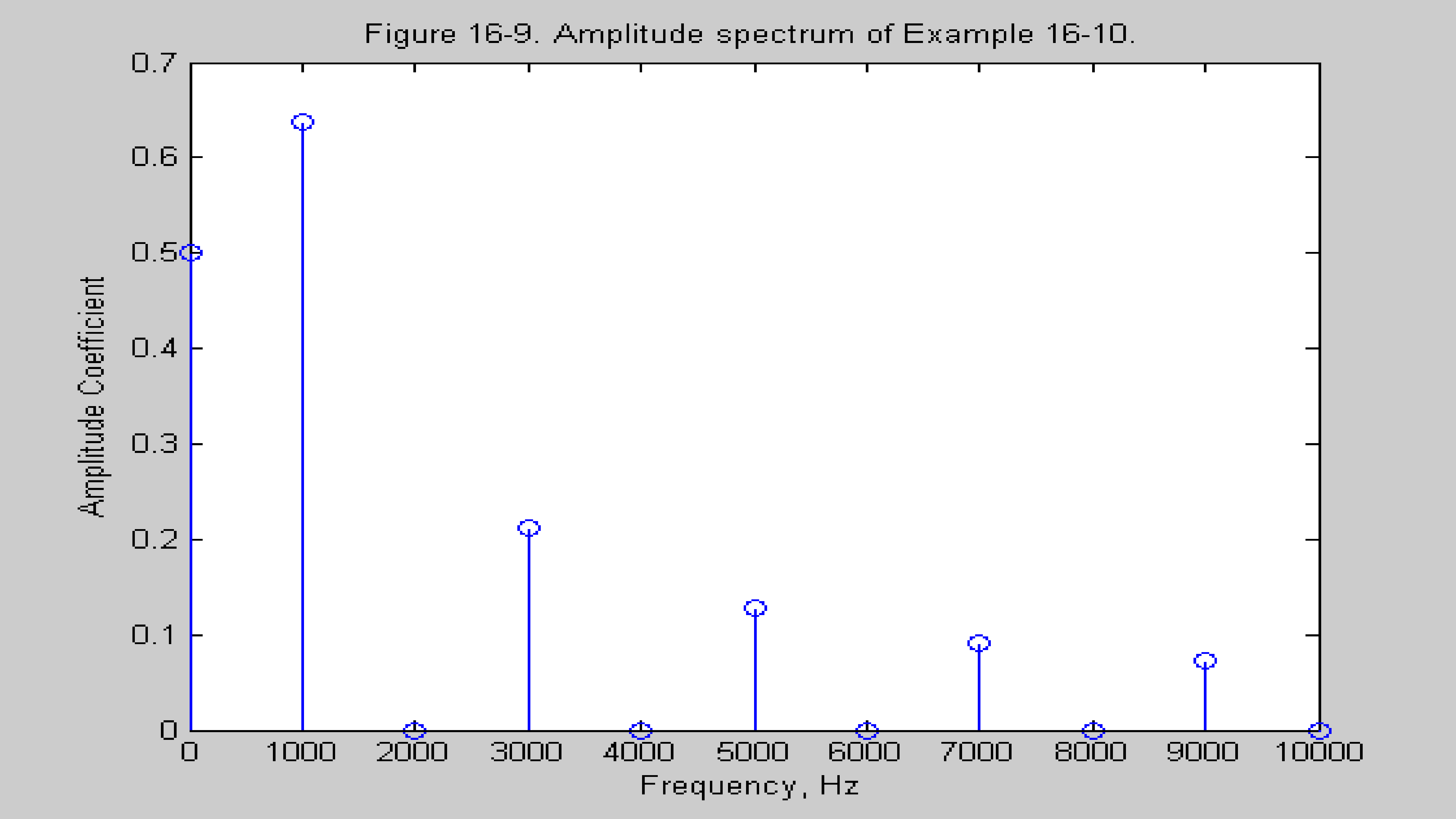


Figure 16-9. Amplitude spectrum of Example 16-10.
Amplitude Coefficient
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
1000
2000
3000
4000
5000
6000
7000
8000
9000
10000
Frequency, Hz

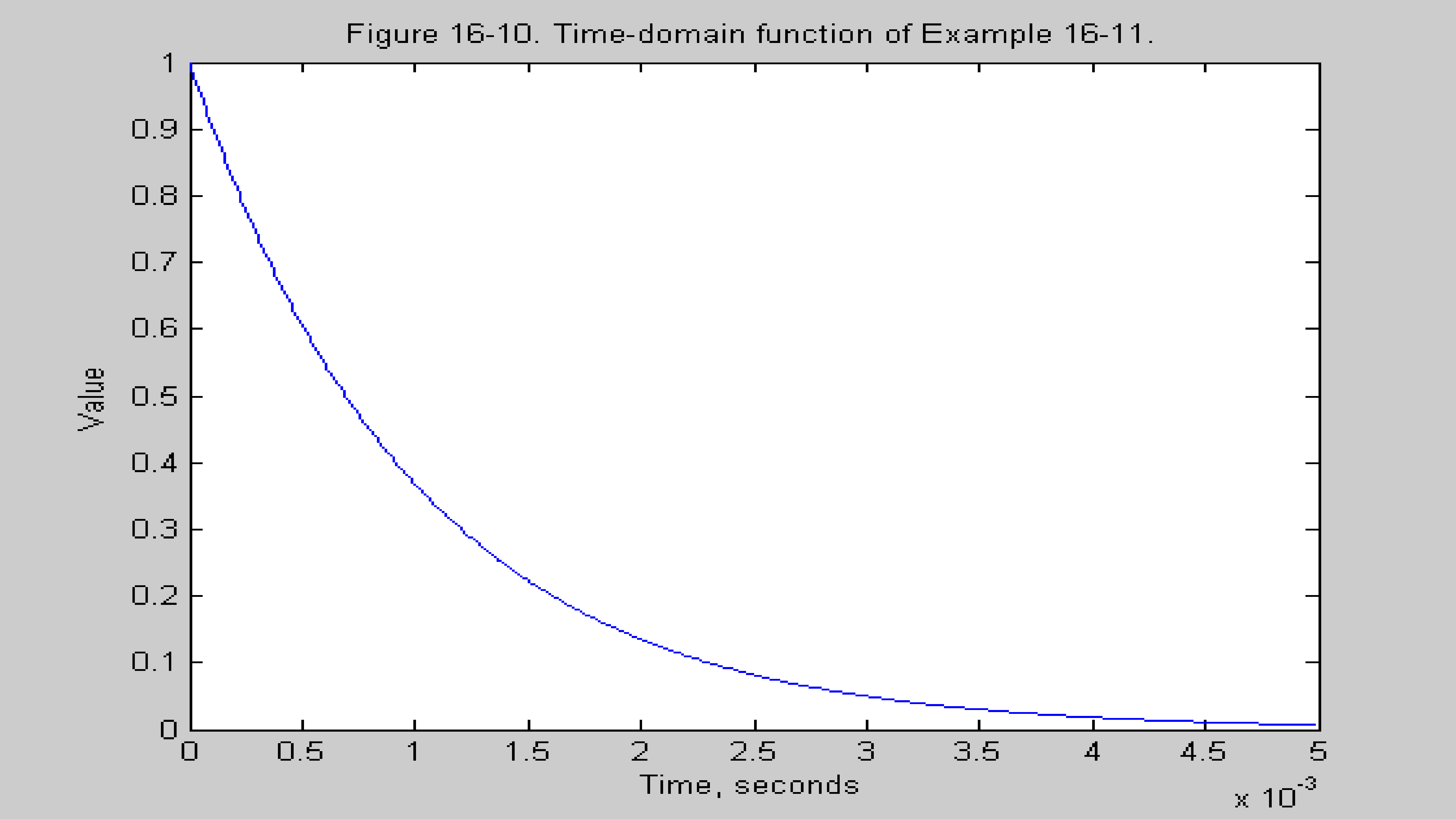
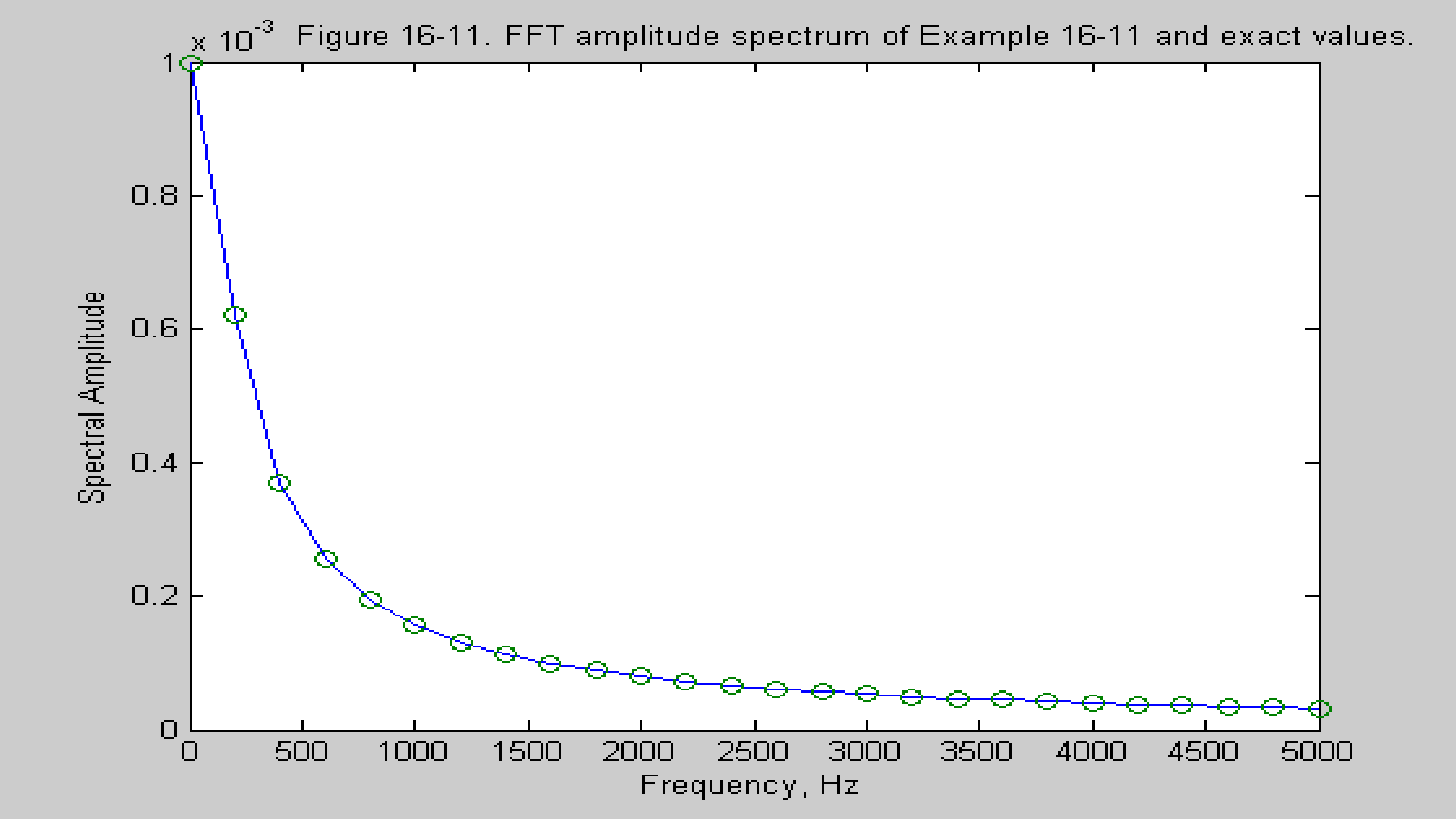


Figure 16-10. Time-domain function of Example 16-11.
Value
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5
Time, seconds
x 10-3


x 10^-3
Figure 16-11. FFT amplitude spectrum of Example 16-11 and exact values.
Spectral Amplitude
Frequency, Hz
1
0.8
0.6
0.4
0.2
0
0
500
1000
1500
2000
2500
3000
3500
4000
4500
5000

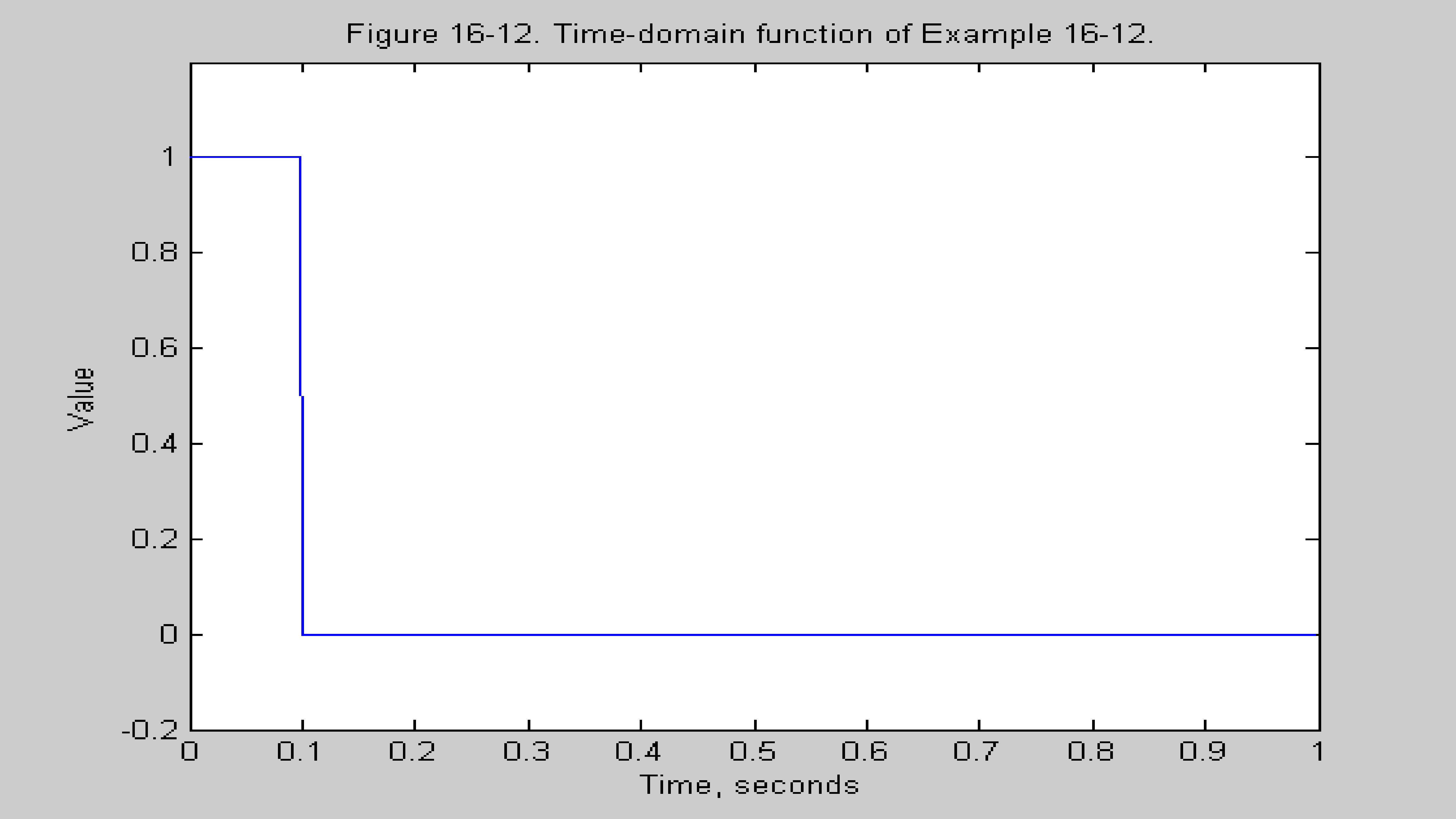


Figure 16-12. Time-domain function of Example 16-12.
Value
1
0.8
0.6
0.4
0.2
0
-0.2
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
Time, seconds

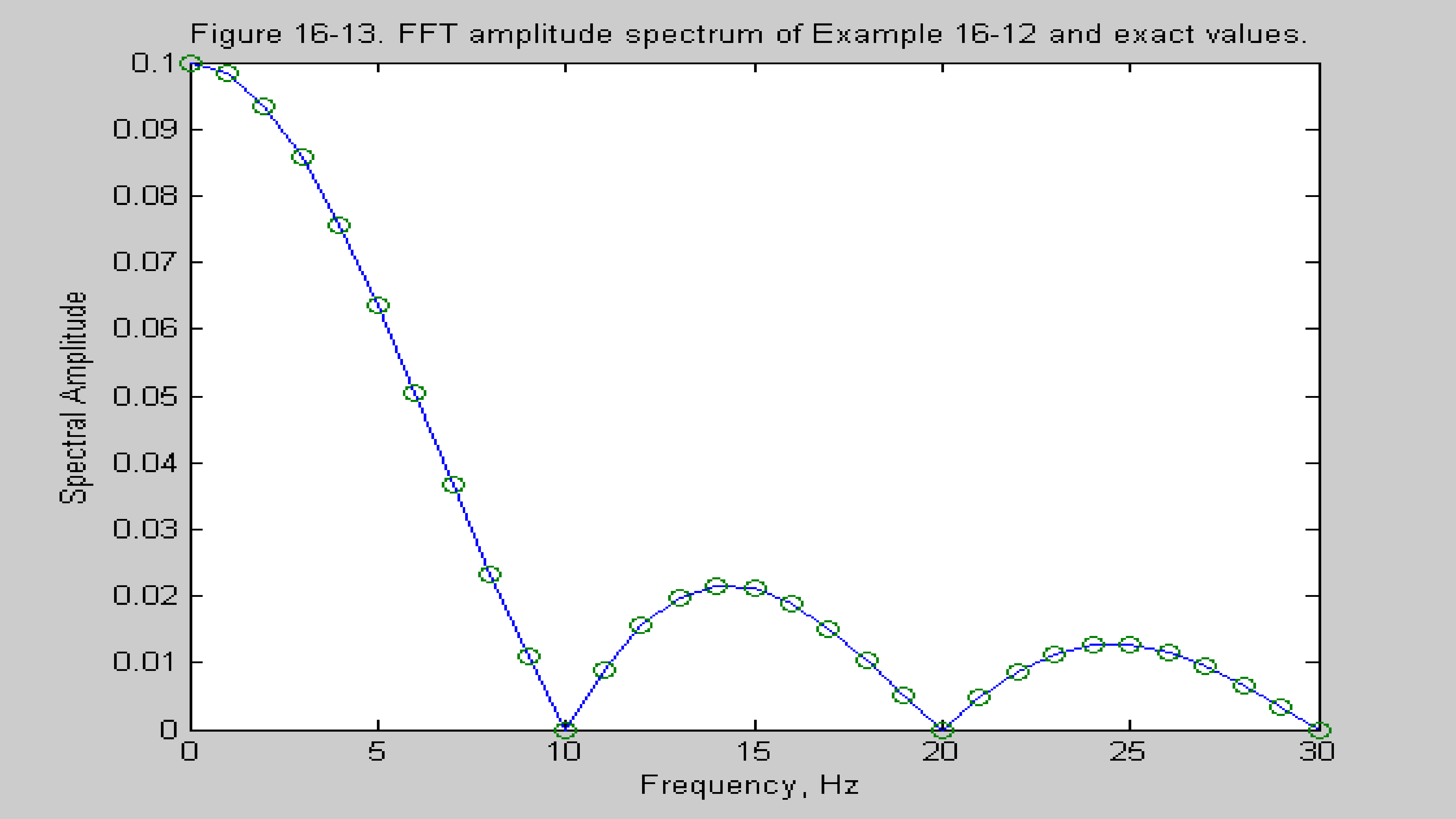


Figure 16-13. FFT amplitude spectrum of Example 16-12 and exact values.
Spectral Amplitude
0.1
0.09
0.08
0.07
0.06
0.05
0.04
0.03
0.02
0.01
0
0
5
10
15
20
25
30
Frequency, Hz